EE708 Seminar
Communication with Noiseless Feedback
(Schalkwijk and Kailath 1966)

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Overview

Problem Formulation	Encoding	Error Analysis	Extensions
• Background • Motivation • Feedback Channels	• Robbins-Munro Procedure • RM in feedback channels	• Channel Capacity • Proof of Channel Capacity	• Bandwidth Limited Signals • Noisy Feedback Links

Channels without feedback

Background

- For continuous time channels, regular-simplex-based coding is popular
- Each of the M continuous-time equal-energy signals have low cross-correlation (zero for orthogonal codes in case of large M)
- For decoding, use a bank of M correlation detectors whose outputs are scanned
- Argmax is the decoded message

Analysis

Background

- For channels without feedback, the below holds for large M (from [5])
- Here, T: time duration of signal, C: Capacity, R: Rate

$$P_{e,\ \mathrm{orth}}\ (M, T) = \frac{\mathrm{constant}}{T^{\beta}} e^{-TE(R)}$$

$$E(R) = \left\{ \begin{array}{ll} C/2 - R, & 0 \le R \le C/4 \\ (\sqrt{C} - \sqrt{R})^2, & \frac{C}{4} \le R \le C \end{array} \right\}$$

$$1 \le \beta \le 2$$

- $P_e = 10^{-7}$, C= 1 bit/second, R = 0.8C
- $T_{orth} = 2030$ whereas $T_{fb} = 15$

Motivation

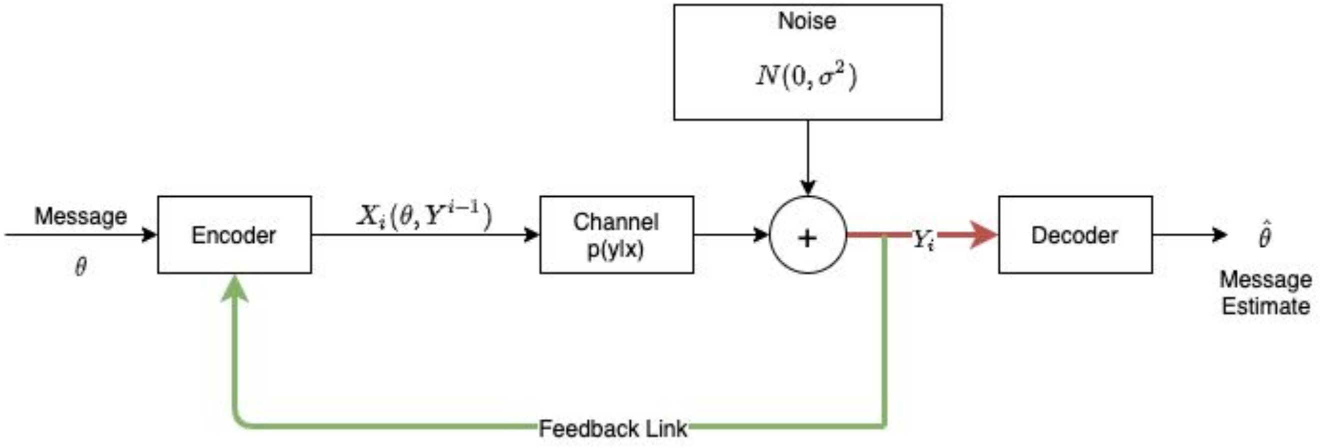
Problem Formulation

- How can feedback help?
  - Simple encoding and decoding schemes
  - Double-exponential decrease in probability of error

- An example: In Space Communication:
  - Satellite -> Ground is noisy due to power constraints
  - Ground -> Satellite can be assumed to be noiseless since power can be very high!

Feedback Channels

Problem Formulation



- Channel Capacity is **unchanged** even in the presence of feedback (Shannon)
- Assumptions:
  - Transmitter -> Receiver link is noisy (Red)
  - Receiver -> Transmitter feedback link is perfect (Green)

Robbins-Munro Algorithm (RM)

Encoding

Goal: Obtain x^* s.t. $f(x^*) = \alpha$ for a given function f and scalar α

Assumption: Function evaluation is noisy i.e. $y_n = f(x_n) + z_n$ where z_n is some additive zero-mean noise
e.g. $z_n \sim \text{Normal}(0, \sigma^2)$

Algorithm: $x_{n+1} = x_n - k_n (y_n - \alpha)$; x_0 is a random initial guess

For convergence:

- $k_n > 0$, $\sum k_n = \infty$ and $\sum k_n^2$ is finite
- f(x) is non-decreasing
- $f'(x^*)$ exists and is positive
- Refer to [3] for more details

Example of RM

Encoding

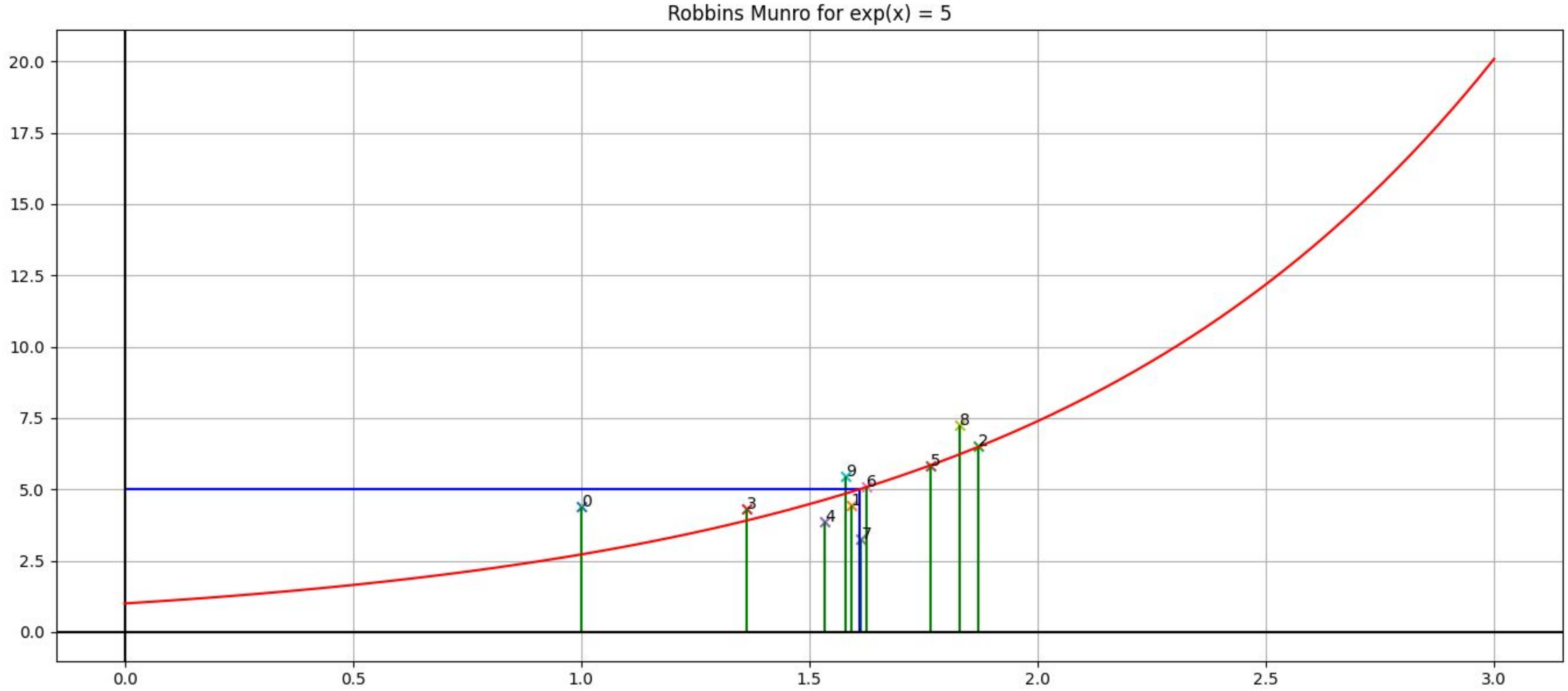
- $f(x) = \exp(x)$
- $\alpha = 5$
- $k_n = 1/n$
- $z_i \sim \text{Normal}(0, 1)$
- Initial guess: $x_0 = 1$

n	x_n	$f(x_n)$	z_n	$\Delta x = -k_n (y_n - \alpha)$
0	1	2.718	1.691	0.591
1	1.591	4.91	-0.466	0.278
2	1.869	6.484	0.033	-0.506
3	1.364	3.911	0.408	0.17
4	1.534	4.637	-0.789	0.23
5	1.764	5.838	0.002	...

After 10 iterations, $|x^* - x_{10}| = |\log(5) - 1.58| = 0.029$

Plot

Encoding



RM in our problem

Encoding

Think of the:

- Transmitter: Function Evaluator/Environment which gives noisy evaluation
- Receiver: Agent who wishes to find solution to $f(x) = \alpha$

Procedure:

1. Transmitter has to send one of M possible messages
2. Divide the unit interval into M disjoint equal-length intervals
3. Choose midpoint of each interval as the message point θ

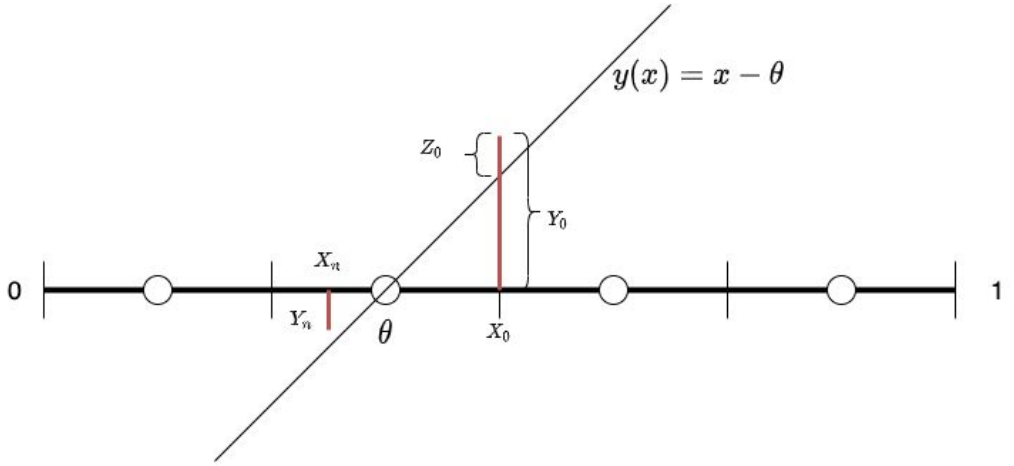
Set α to 0 and $f(x) = m(x - \theta)$ where:

- θ is the message to be sent
- m is some positive value

Example

Encoding

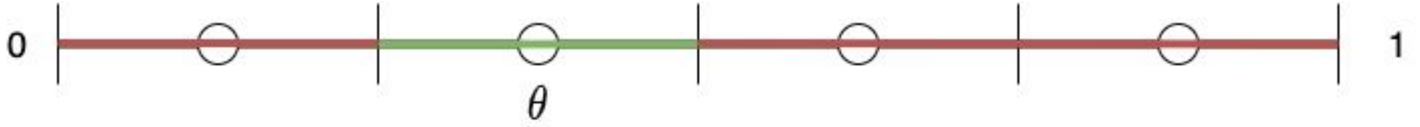
- $M = 4 \rightarrow \theta \in \{0, 1, 2, 3\}$
- To send: $1 \rightarrow \theta = 0.375$
- $f(x) = x - \theta$



After N iterations:

If $X_N \in$ {green region}: successful decoding

Else: incorrect decoding



Channel Capacity

Error Analysis

- Achieves capacity for gaussian channels without bandwidth constraint
  - $C = P_{av}/N_0$ nats/second

- It can be shown the same maximum rate for all additive iid noise channels (not only gaussian)

- $$P_{e,fb} = \frac{1}{\sqrt{6\pi \frac{C}{R} e^{2(C-R)T}}} \exp[-\frac{3C}{2R} e^{2(C-R)T}]$$

Gaussian Channel Capacity - Proof Sketch

Error Analysis

1. Prove that the received message is gaussian distributed with variance inversely proportional to N (N is message size/no of iterations of RM)
2. Thus obtain the formula for rate R = ln M / T = ln $N^{0.5}$/T
3. Compute the expression for average power on N transmissions in terms of time interval of transmission T (since it is a continuous time channel)
4. Substitute T = ln N / 2R and obtain bound on R.

Step 1

Error Analysis

$$X_{n+1} = X_n - \frac{1}{\alpha n} Y_n(X_n)$$

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Feedback

Transmission

Step 1

Error Analysis

$$X_{n+1} = X_n - \frac{1}{\alpha n} Y_n(X_n)$$

Feedback

Transmission

$$\text{Substituting } Y_n(X_n) = \alpha(X_n - \theta) + Z_n$$

Gaussian Noise

Step 1

Error Analysis

Feedback

$$X_{n+1} = X_n - \frac{1}{\alpha n} Y_n(X_n)$$

Transmission

Substituting $Y_n(X_n) = \alpha(X_n - \theta) + Z_n$

Gaussian Noise

$$X_{n+1} = (\frac{n-1}{n}) X_n + \frac{\theta}{n} - \frac{Z_n}{\alpha n}$$

$$n X_{n+1} = (n-1) X_n + \theta - \frac{Z_n}{\alpha}$$

Step 1

Error Analysis

$$\sum_{i=2}^{n+1}(i-1)X_i = \sum_{i=1}^{n}((i-1)X_i + \theta - \frac{Z_i}{\alpha})$$

$$nX_{n+1} = n\theta - \sum_{i=1}^{n}\frac{Z_i}{\alpha}$$

$$X_{n+1} = \theta - \frac{\sum_{i=1}^{n} Z_i}{n\alpha}$$

$$X_{n+1} \sim \mathcal{N}(\theta, \frac{\sigma^2}{\alpha^2 n})$$

Step 1

Error Analysis

$$\sum_{i=2}^{n+1}(i-1)X_i = \sum_{i=1}^{n}((i-1)X_i + \theta - \frac{Z_i}{\alpha})$$

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$$X_{n+1} = \theta - \frac{\sum_{i=1}^{n} Z_i}{n\alpha}$$

$$X_{n+1} \sim \mathcal{N}(\theta, \frac{\sigma^2}{\alpha^2 n})$$

P_e ➡ 0 as n ➡ ∞

Step 2

Error Analysis

To decode a message correctly, we need X_{n+1} to be in the range $\theta \pm \frac{1}{2M}$

$$\text{Hence } P_e = 2\text{erfc}\left(\frac{\frac{1}{2}M^{-1}}{\sigma/\alpha\sqrt{N}}\right)$$

$$\text{For } P_e \to 0,\ M(N) = N^{\frac{1}{2}-\epsilon}$$

$$\text{Also, } R = \frac{\log M(N)}{T}$$

$$\text{As } \epsilon \to 0,\ R \to C \implies C = \frac{\log N}{2T}$$

Steps 3 & 4

Error Analysis

$$P_{av}(N) = \frac{1}{T} E\left[\alpha^2(X_1 - \theta)^2 + \sum_{i=1}^{N-1} \alpha^2(X_{i+1} - \theta)^2\right]$$

$$P_{av}(N) = \frac{2C}{\log N}\left(\frac{\alpha^2}{12} + \sigma^2 \sum_{i=1}^{N-1} \frac{1}{i}\right)$$

$$\lim_{n \to \infty} P_{av}(N) = 2\sigma^2 C = N_0 C$$

$$C = P_{av}/N_0$$

Probability of error

Error Analysis

- From step 2: $$P_e = 2\text{erfc}\left(\frac{\frac{1}{2}M^{-1}}{\sigma/\alpha\sqrt{N}}\right)$$

- Using a well-known asymptotic formula for *erfc* and substituting the value for M and optimal α (after differentiation):

$$P_e \approx \frac{\exp\left[-\frac{3}{2}\left(\frac{R}{C}\right)^{-1}N^{\epsilon}\right]}{\left[6\pi\left(\frac{R}{C}\right)^{-1}N^{\epsilon}\right]^{1/2}}$$

- $N = e^{2CT}$ and $R = (1 - \varepsilon)C$ ...(by definition)

- Hence, we finally get:

$$P_e \approx \frac{\exp\left[-\frac{3C}{2R}e^{2(C-R)T}\right]}{\left[6\pi\frac{C}{R}e^{2(C-R)T}\right]^{1/2}}$$

Extensions & Additional Discussion

Extensions

- Band Limited Signals
  - Achieved a similar scheme that attains channel capacity for bandlimited signals
- Noise in feedback link
  - In the presence of feedback noise, capacity is not attained.
  - To enforce 0 error for large N, rate tends to 0.
- Non Gaussian Additive Noise
  - The proof still holds, but to prove X_n is gaussian asymptotically Sack's theorem is required

Key Takeaways

- Noiseless Wideband Feedback Channel Coding Schemes
  - Simplify encoding & decoding
  - Practically occurring scenario such as ground-space communication
- Schalwijk & Kailath Encoding Scheme
  - Based on Robbins-Munro Procedure
  - Achieves optimal channel capacity for gaussian channels
  - General scheme can be applied to other additive noise models
  - Probability of Error decays faster as a function of T when compared with orthogonal coding

Thank You

References

[1] A coding scheme for additive noise channels with feedback--I: No bandwidth constraint - Authors J. Schalkwijk, T. Kailath 1966

[2] https://www2.isye.gatech.edu/~yxie77/ece587/Lecture16.pdf

[3] https://en.wikipedia.org/wiki/Stochastic_approximation#Robbins–Monro_algorithm

[4] A coding scheme for additive noise channels with feedback--II: Band-limited signals - Author J. Schalkwijk 1966

[5] L. H. Zetterberg, "Data transmission over a noisy Gaussian channel"