Auction Theory

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Auction Theory

- Definition:

  The branch of economic theory dealing with auction types and participants' behavior in auctions

- Perspective and Approach:
  - Game Theory - Auctions are Bayesian games of incomplete information
  - Contract or Mechanism Design Theory - Auctions are allocation mechanisms
  - Market Microstructure - Auctions are models of price formation
  - Context of different applications - procurement, patent licensing, public finance
- The Game Theoretic approach is used in this presentation

The Independent Private Value Model

- Auction environment and setup
  - Bidders $i = 1,...,n$
  - One object to be sold
  - Bidder *i* observes a "signal" $S_i \sim F(\cdot)$, with typical realization $s_i \in [\underline{s}, \overline{s}]$ and assume F is continuous
  - Bidders' signals $S1, ..., Sn$ are independent (and <u>private</u>)
  - Bidder i's value $v_i(s_i) = s_i$.
  - Payoff = value – amount paid if won, else zero.

Second Price Sealed Bid Auction – Vickrey Auction

- Vickrey Auction setup
  - Bidders are asked to submit sealed bids $b_1,\ldots,b_n$.
  - Bidder who submits the highest bid is awarded the object
  - Bidder pays the amount of the second highest bid
- Revenue
  - Each bidder bids their value
  - The seller's revenue (the amount paid in equilibrium) = second highest value.
  - Let $S^{i:n}$ denote the ith highest of n draws from distribution F
  - ($S^{i:n}$ is a random variable with typical realization $s^{i:n}$)
  - Seller's expected revenue = $E[S^{2:n}]$

Second Price Sealed Bid Auction – Vickrey Auction

THEOREM: In a second price auction, it is a weakly dominant strategy to bid one's value, $b_i(s_i) = s_i$.

Suppose *i*'s value is s_i, and she considers bidding $b_i > s_i$.

Let denote the highest bid of the other bidders . From *i*'s perspective, this is a random variable.

There are three possible outcomes from *i*'s perspective:

(i) $> b_i, s_i$ (ii) $b_i >$ $> s_i$ or (iii) b_i , $s_i >$.

In outcome (i) or (iii), *i* would have done equally well to bid s_i rather than $b_i > s_i$.

In (i) she won't win regardless, and

In (ii) she will win, and will pay regardless.

However, in case (ii), *I* will win and pay more than her value if she bids , something that won't happen if she bids s_i.

Thus, *i* does better to bid s_i than $b_i > s_i$.

A similar argument shows that *i* also does better to bid s_i than to bid $b_i < s_i$.

Q.E.D.

First Price Sealed Bid Auction

- Auction environment and setup
  - –Bidders submit sealed bids $b_1,...,b_n$.
  - –Bidders who submits the highest bid is awarded the object
  - –Bidder pays his bid.
- Value realization
  - –Bidders will not want to bid their true values
  - –Bidding true value implies a zero profit.
  - –By bidding somewhat below their values, they can potentially make a profit some of the time.

First Price Sealed Bid Auction

- Look for a Symmetric Equilibrium; assume strategy is a strictly increasing, differentiable function of his value
- Bidder *i*'s expected payoff, as a function of his bid b_i and signal s_i is

$$U(b_i, s_i) = (s_i - b_i) \,.\, \Pr\left[b_j = b\left(S_j\right) \leq b_i,\ \forall_j \neq i\right]$$

- Where $b_j = b(s_j)$ is his strategy function. Bidder *i* chooses to solve

$$\max_{b_i}\left(s_i - b_i\right)\ F^{n-1}\left(b^{-1}\left(b_i\right)\right)$$

- First order condition is

$$\left(s_i - b_i\right)(n-1)F^{n-2}\left(b^{-1}\left(b_i\right)\right)f(b^{-1}\left(b_i\right))\frac{1}{b'\left(b^{-1}\left(b_i\right)\right)} - F^{n-1}\left(b^{-1}\left(b_i\right)\right) = 0$$

First Price Sealed Bid Auction

- At a symmetric equilibrium $b_i = b(s_i)$, the first order condition reduces to a differential equation

$$b'(s) = (s - b(s))(n-1)\frac{f(s)}{F(s)}$$

- This can be solved using the boundary condition $b(\underline{s}) = \underline{s}$ to obtain

$$b(s) = s - \frac{\int_{\underline{s}}^{s_i} F^{n-1}(\tilde{s})d\tilde{s}}{F^{n-1}(s)}$$

- This agrees with the assumption that b(s) is increasing and differentiable. Hence any symmetric equilibrium with these properties must involve bidders using the strategy b(s)

Revenue Equivalence Theorem

Suppose bidders have values $s_1, s_2, \ldots, s_n$ identically and independently distributed with CDF $F(\,.\,)$. Then, all auction mechanisms that:
(i) always award the object to the bidder with highest value in equilibrium
(ii) give a bidder with valuation $\underline{s}$ zero profits
generates the **same revenue in expectation**

$\mathbb{E}$[Revenue] = $\mathbb{E}[S^{2:n}]$ = expectation of second highest value
(Expectation is over bidding strategy of each player and hence expected revenue is independent of the bidding strategies!)

Revenue Equivalence Theorem

As long as an auction satisfies the two conditions,

- Calculation of equilibrium strategy $b(s)$ is simple
- The seller is assured of a constant expected revenue

Useful Properties:

- In any symmetric equilibrium, each bidder must use a continuous and strictly increasing strategy
- Equilibrium payoff $U(s_i) = \int_{\underline{s}}^{s_i} F^{n-1}(x)dx$ (From Envelope Theorem)

Application: All-Pay Auction

- Same setup as that of previous: $s_1, s_2, \ldots, s_n$ are i.i.d.
- Bidders submit bids $b_1, b_2, \ldots, b_n$ and the bidder who submits the highest bid wins
- However, bidders must pay their bid **regardless** of whether they win the auction.
- Let $b^A(s)$ be the increasing strategy used by all players (symmetric equilibrium)

$$U(s_i) = s_i F^{n-1}(s_i) - b^A(s_i) = \int_{\underline{s}}^{s_i} F^{n-1}(\tilde{s}) d\tilde{s}$$

Which implies:

$$b^A(s_i) = s_i F^{n-1}(s_i) - \int_{\underline{s}}^{s_i} F^{n-1}(\tilde{s}) d\tilde{s}$$

Application: English Auction

- Oral Ascending Auction
- All bidders start in the auction with a price of zero
- The price rises continuously, and bidders may drop out at any point in time. Once they drop out, they cannot re-enter
- The auction ends when only one bidder is left, and this bidder pays the price at which the second-to-last bidder dropped out

- Point 4 $\Longrightarrow$ (i) satisfied
- Bidder with value $\underline{s}$ will be the first to drop out $\Longrightarrow$ (ii) satisfied

Application: Dutch Auction

- Descending Price Auction
- The price starts at a very high level and drops continuously
- At any point in time, a bidder can stop the auction, pay the current price and end the auction
- Similar to a First Price Sealed Bid auction

- Point 2 $\Longrightarrow$ (i) satisfied
- Bidder with value $\bar{s}$ can never win $\Longrightarrow$ (ii) satisfied

Common Value Auctions

Generalise the model:

(i) learning bidder j's information could cause bidder i to re-assess his estimate of how much he values the object
i.e. Value to bidder i is $v_i(s_i, s_{-i}) \neq s_i$ in general

(ii) the information of i and j is not independent i.e. S_i and S_j could be correlated
e.g. when j's estimate is high, i's is also likely to be high

Can be framed mathematically …

Example of Common Value Auction

- An auction for a natural resource like a tract of timber
- Bidders are likely to have different costs of harvesting the timber
- Bidders are likely to be unsure about how the quantity of harvestable timber and use some sort of statistical sampling to estimate it
- These estimates will be based on limited sampling $\Longrightarrow$ they will be imperfect – so if i learned that j had sampled a different area and got a low estimate, she would likely revise her opinion of the tract's value
- If the areas sampled overlap, estimates are unlikely to be independent

Special cases

Independent Private Model:

- $S_1, S_2, \ldots, S_n$ are independent and $v(s_i, s_{-i}) = s_i$

Pure Common Value model with conditionally independent signals:

- All bidders have the same value given by a random variable V
- $S_1, S_2, \ldots, S_n$ are each correlated with V but independent conditional on it e.g. $S_i = V + \epsilon_i$ where $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ are independent
- $v(s_i, s_{-i}) = \mathbb{E}[V \mid s_1, s_2, \ldots, s_n]$
- Timber auction is an example (if harvesting costs are the same)

Multiple Auctions

- Setting:

  More than one identical item to be sold, but each bidder has use for at most one.
- Variations on progressive multiple auctions:
  - **Simultaneous Multiple Auction** - applicable when items are identical
  - **Successive Multiple Auction –** minor variation in quality of items exists
- Mutiple Auctions by Sealed Bids

Simultaneous and Successive Multiple Auction

- **Simultaneous Multiple Auction**
  - Each bidder permitted to raise bid even if it does not make it highest
  - When no bidder wants to raise bid, items awarded to M highest bidders (M items)
  - Pareto Optimal Result: Bidders with top M values for the item, secure it at value of (M+1)th person
- **Successive Multiple Auction**
  - Each item auctioned successively
  - Each bidder must consider whether he should bid higher on current item, or sign off hoping he will get lower price on next item
  - Characteristics similar to Dutch auction

A Simple Set Up

- Assume two identical items, N bidders, simultaneous auctioning
- Assumed that first auction is by sealed bids, price = second highest bid
- Bidders are similar:
  - Values drawn from distribution common to all bidders
  - Function mapping values to bids common to all bidders, say x(v)
- Assumed that ordering between values and bids is same

Equilibrium Analysis

- Suppose a bidder having drawn value v, contemplating a deviation by raising bid from x(v) to x(v)+dx
- Affects outcome when highest of other N-1 bids lies between x(v) to x(v)+dx
- This increased bid secures first item at a price between x(v) and x(v)+dx
- Price paid for second item = Highest of the (N-2) values drawn by unsuccessful bidders in second auction
- When uniform distribution in [0,1], expected value of price for second item = [(N-2)/(N-1)] v
- Since if increment of dx being considered causes any change in the outcome, none of these (N-2) unsuccessful bidders can have drawn a value greater than v, or there would be expected gain from increasing the bid

Equilibrium Analysis

- Thus, equilibrium situation, each bidder puts in a bid of
  - Bi = [(N-2)/(N-1)] Vi
- Average of second highest value = (N-1)/(N+1)
- Second highest bid = Price of first item = (N-2)/(N+1)
- The above is also expected price for second item

- **Analysis of Successive Auction**
  - Slightly different dispersion of prices
  - Complications of complete analysis too formidable

Optimal Auction Design

- Problem Description:
  - Person has an object to sell
  - Does not know how much prospective buyers are willing to pay
  - Wants to maximise his expected revenue by selling the object

- Auctions studies as non-cooperative games with imperfect information

Optimal Auction Design – Assumptions and Feasibility Conditions

- **Assumptions:**
  - Seller's uncertainty modeled as a probability distribution of value estimate for bidder I over a finite interval
  - Seller's value estimate of the item for auction is known to all bidders
  - Preference uncertainty (does not affect other bidders' estimates) and Quality uncertainty (affects other bidders' estimates)
  - Introduces **Revision Effect Functions** for allowing for Quality Uncertainty

- **Feasibility Conditions:**
  - <u>Individual Rationality Condition:</u> Seller can't force a bidder to participate in an auction which offers him less expected utility than he could get on his own. Therefore, expected utility for each player >= 0.
  - <u>Incentive Compatibility Condition:</u> Seller can't prevent any bidder from lying about his estimates if he expects to gain by lying. To ensure no bidder has incentive to lie about his value estimate, we impose this condition (Mathematical)

Optimal Auction Design – Key Principle

Definition: Direct Revelation Mechanism

In a direct revelation mechanism, the bidders simultaneously and confidentially announce their value estimates to the seller; and the seller then determines who gets the object and how much each bidder must pay, as some functions of the vector of
announced value estimates.

Lemma: The Revelation Principle

Given any feasible auction mechanism, there exists an equivalent feasible direct revelation mechanism which gives to the seller and all bidders the same expected utilities as in the given mechanism.

- This enables us to solve auction design problems as there is no loss of generality in considering only direct revelation mechanism, which are comparatively easier to solve

Thank You

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